SOME SOLUTIONS TO EQUATIONS OF MOTION IN THE EQUATORIAL REGIONS

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PREFACE

This investigation was conducted at the United States Naval Postgraduate School, Monterey, California as the thesis requirements for the degree of Master of Science in Aerology.

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TABLE OF SYMBOLS AND ABBREVIATIONS

λ	Coriolis parameter = $2\omega \sin\phi$
ρ	Density of air
t	Time
ϕ	Latitude
ω	Angular speed of the earth = 7.29×10^{-5} sec^{-1}
x	Coordinate axis - East
y	Coordinate axis - North
u	Easterly velocity component
v	Northerly velocity component
a	Radius of the earth = 6378 km
T	Period of pressure variation = 12 hours
C_1, C_2	Constant of integration

I. INTRODUCTION

In mid-latitudes the geostrophic and gradient solutions of the equations of motion generally give a good approximation to the true wind. However, in the equatorial regions the acceleration terms, both centripetal and tangential, are frequently of the same order of magnitude as the pressure and Coriolis forces; hence the gradient and geostrophic solutions are not generally applicable.

To render the equations integrable various simplifications have been used. Notable among these are the assumptions that the flow is (a) horizontal, (b) frictionless, (c) homogeneous, (d) non-divergent and (e) steady. Some investigators have neglected the Coriolis force, while others have either treated it as a constant or replaced $\sin \phi$ by ϕ. Practically all authors assume (a) through (c) above as will be done by all discussions in this paper.

In addition to the above assumptions, Grimes [5] let motion be independent of longitude. He allowed for variation of the Coriolis force by assuming $\sin \phi = \phi$ with ϕ as a function of y. His solutions gave particular streamlines which were determined from initial values of velocity and vorticity. Pressure distributions were also determined from these solutions; however, it is generally preferable to derive motions from known pressure distributions and initial velocities. Grimes obtained isobaric patterns which were similar to mean maps in the Indian and South Pacific oceans but pressure gradients were less

than those usually observed. Crossley [2] found this solution to apply only to mean motions and not to the instantaneous flow pattern.

Crossley [1] removed the restriction on independence of longitude and corrected the fault that all isobars crossed the equator at right angles in the above solution. The same author [3] obtained a solution in spherical coordinates.

Grimes' original assumptions less the requirement of non-divergence were used in Chapter I of this paper.

Forsdyke [4] took a constant Coriolis force with motion at any instant everywhere the same, that is, the space derivatives do not appear in acceleration terms. He assumed a pressure distribution of the form $p = p_0 + A(t)y$ for various functions $A(t)$.

By expressing the resulting motion as geostrophic plus ageostrophic components, he suggests a method for relating wind to the pressure distribution in the tropics.

In Chapter II an attempt is made to generalize Forsdyke's solution by making less restrictions on the Coriolis parameter and assuming a pressure distribution of the form $p = p_0 + x \sin \frac{2\pi t}{T}$.

In Chapter III a pressure field of the form $p = p_0 + Ae^{-\epsilon t} \quad B(1-e^{-\gamma t})y$ where A, B, ϵ and γ are constants is discussed. This pressure field was suggested but not developed by Forsdyke [4].

More recently, Schmidt [7] concentrated on the kinematics of flow by starting with Rossby's vorticity equation and assuming the dependent variables to be functions of y only. He obtained values for u in terms

of y assuming v known and plotted streamlines for a monsoon current, high pressure at the equator (doldrums), and a line of convergence stretching in an east-west direction along the equator. Some of the streamlines obtained in Chapter II of this paper closely resemble Schmidt's.

Grimes [6] using his original assumptions less independence with longitude, obtains a set of pseudo-geostrophic equations where streamlines are absolute vorticity trajectories. They are parallel to isobars of "dynamic pressure" ($P = p + \frac{1}{2}\rho V^2$, where $V^2 = u^2 + v^2$). The difficulty in this approach is the necessity for drawing accurate isobars of dynamic pressure, which must be drawn at intervals of one-fifth of a millibar. This is particularly difficult where reduction to sea level is necessary.

II. MOTION AS A FUNCTION OF y ALONE

The equations of horizontal frictionless motion are:

$$\frac{du}{dt} - 2\omega v \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{dv}{dt} + 2\omega u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (2)$$

In this solution steady-state is assumed with motion independent of longitude ($\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$). Pressure is a linear function of latitude (y) alone with isobars equally spaced and parallel to the equator. The Coriolis force is treated in a similar manner to Grimes [5] by letting $\sin\phi = \phi = y/a$. Thus the Coriolis parameter is Ky where $K = 2\omega/a$.

With these assumptions the equations of motion become:

$$v\frac{\partial u}{\partial y} = Kyv \qquad (3)$$

$$v\frac{\partial v}{\partial y} = \alpha\frac{\partial p}{\partial y} - Kyu \qquad (4)$$

Equation (3) readily integrates to

$$u = \frac{1}{2}Ky^2 + u_0 \qquad (5)$$

Substitution of this value of u into equation (4) and integrating yields the following expression for v,

$$v^2 = v_o^2 + 2\alpha(p - p_o) - \frac{1}{4} K^2 y^4 - K u_o y^2 \qquad (6)$$

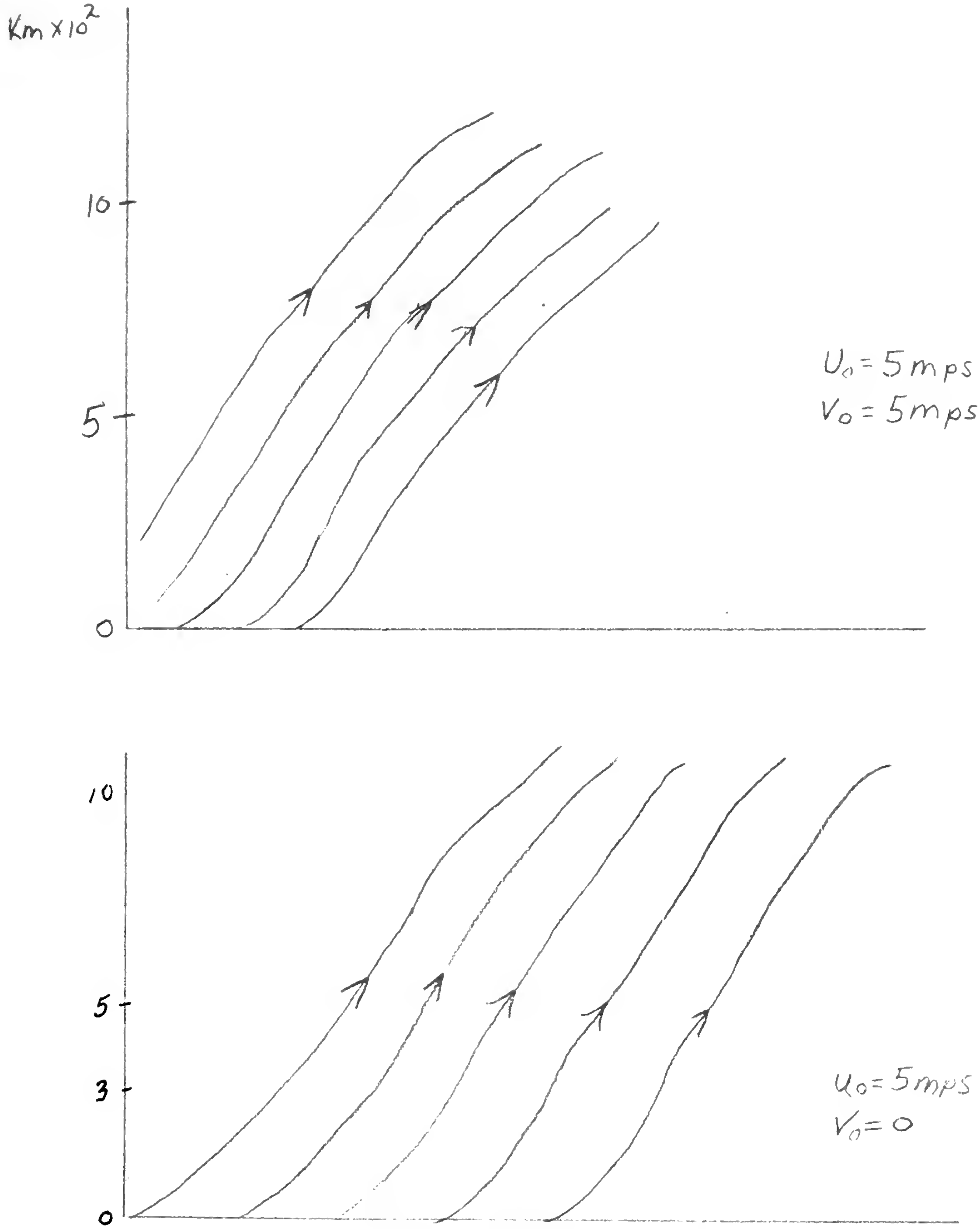
u_o, v_o, p_o being values of velocity and pressure at $y = 0$.

Numerical values of the slope of the streamlines were found at every 500 kilometers with varying initial conditions and a constant pressure gradient of 1.5 millibars per 500 km and a specific volume of 860 cm^3/gm.

These streamline patterns (see Figures 1-5) resemble observed mean motions in the Indian and Southern Pacific oceans. Values of v for $y > 1500$ km were imaginary indicating steady-state is no longer valid at that distance from the equator with the given initial conditions.

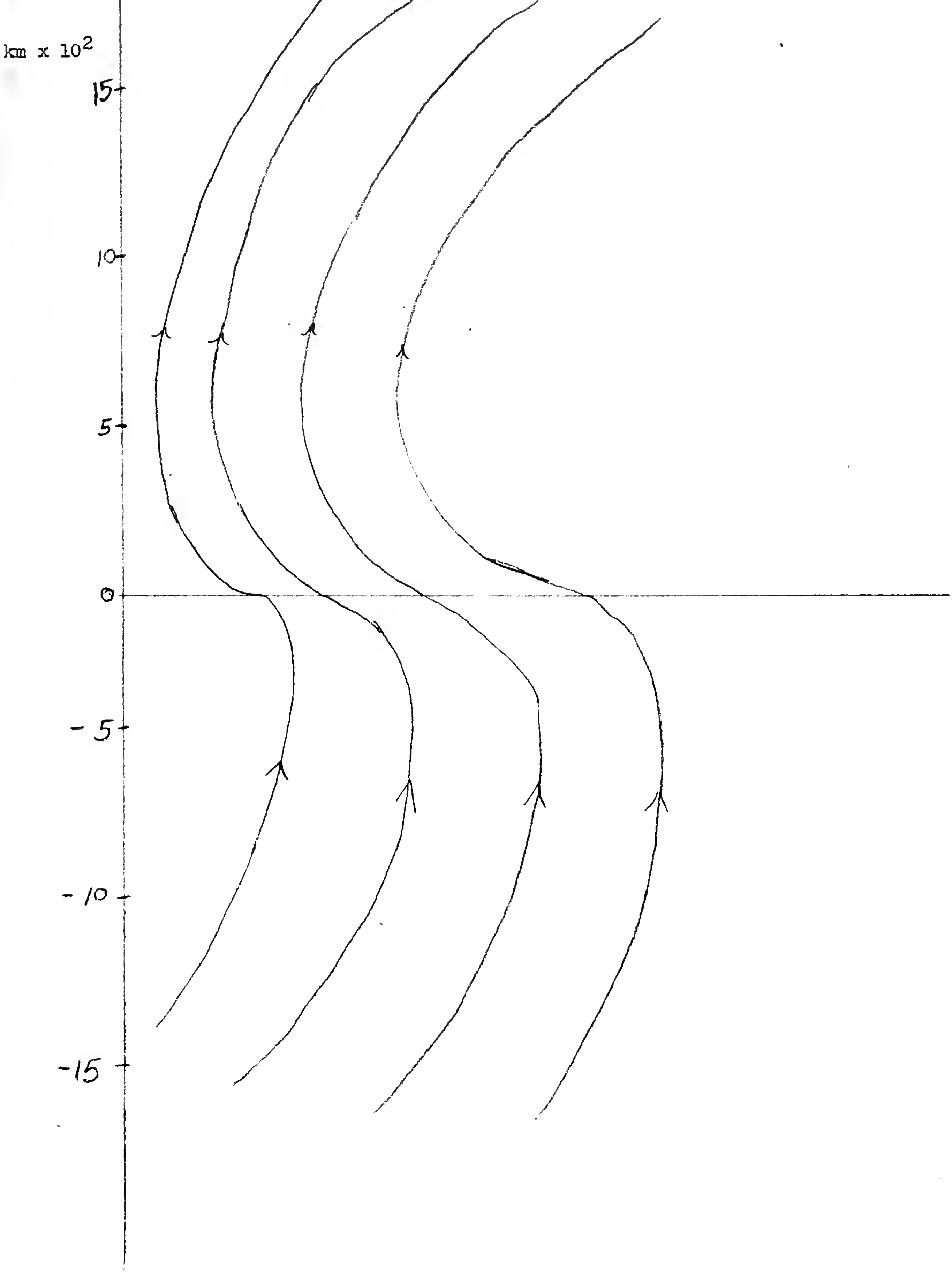
Most of the faults of Grimes' and Crossley's solutions still remain in this simple pattern, the most important being that steady state is not generally applicable and therefore solutions apply only to mean motions. It should also be noted that the determination of the streamline patterns from even these simple pressure distributions is a laborious process.

However, one improvement is the removal of the restriction of no horizontal divergence. Zones of divergence and convergence are known by observations to exist in the equatorial regions.



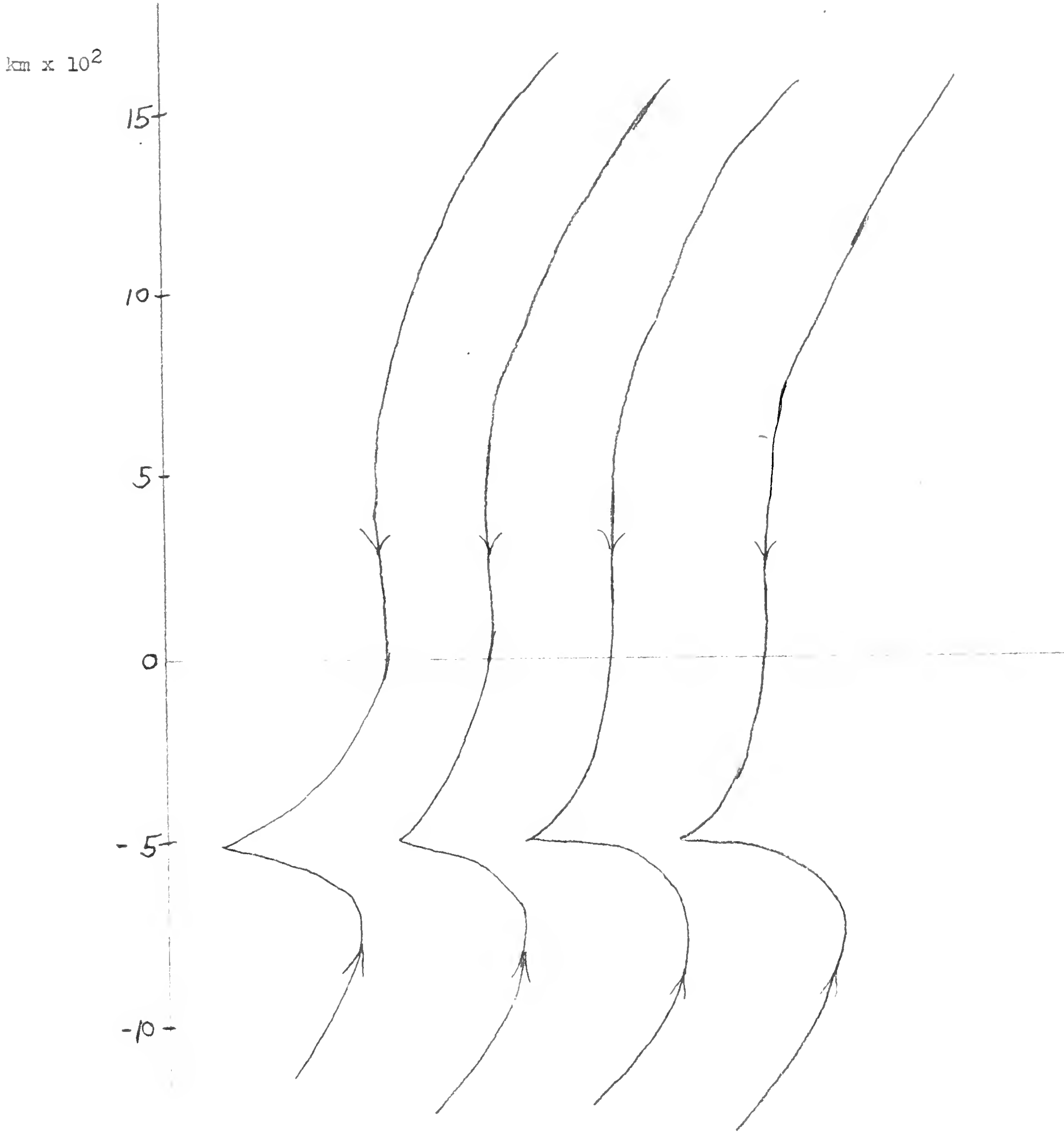
Sea level streamlines for $u_o = 5$ mps, $v_o = 5$ mps and for $u_o = 5$ mps, $v_o = o$ at $y = 0$

Fig. 1



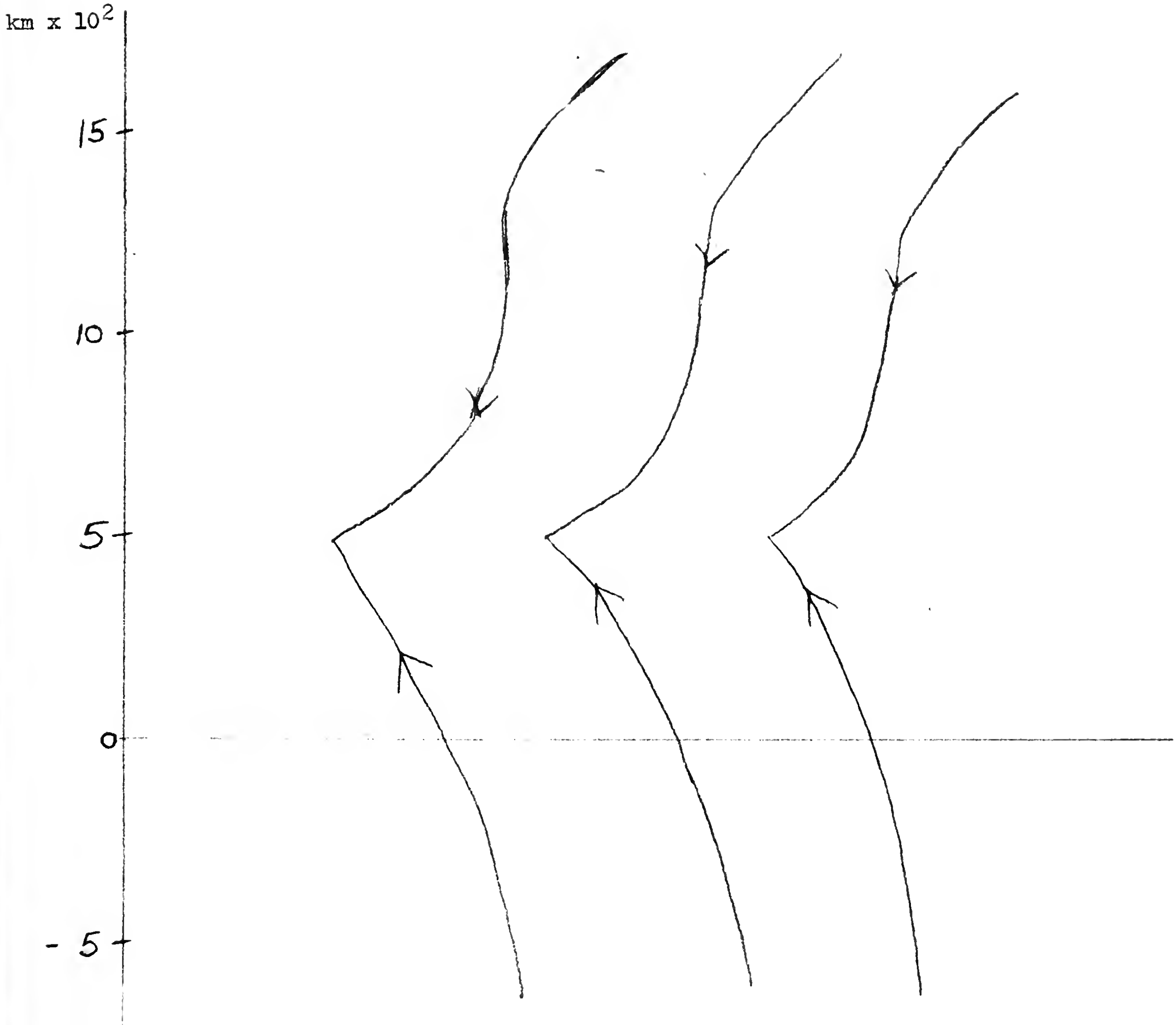
Sea level streamlines for
u_o = -5 mps, v_o = 0 at y = 0

Fig. 2



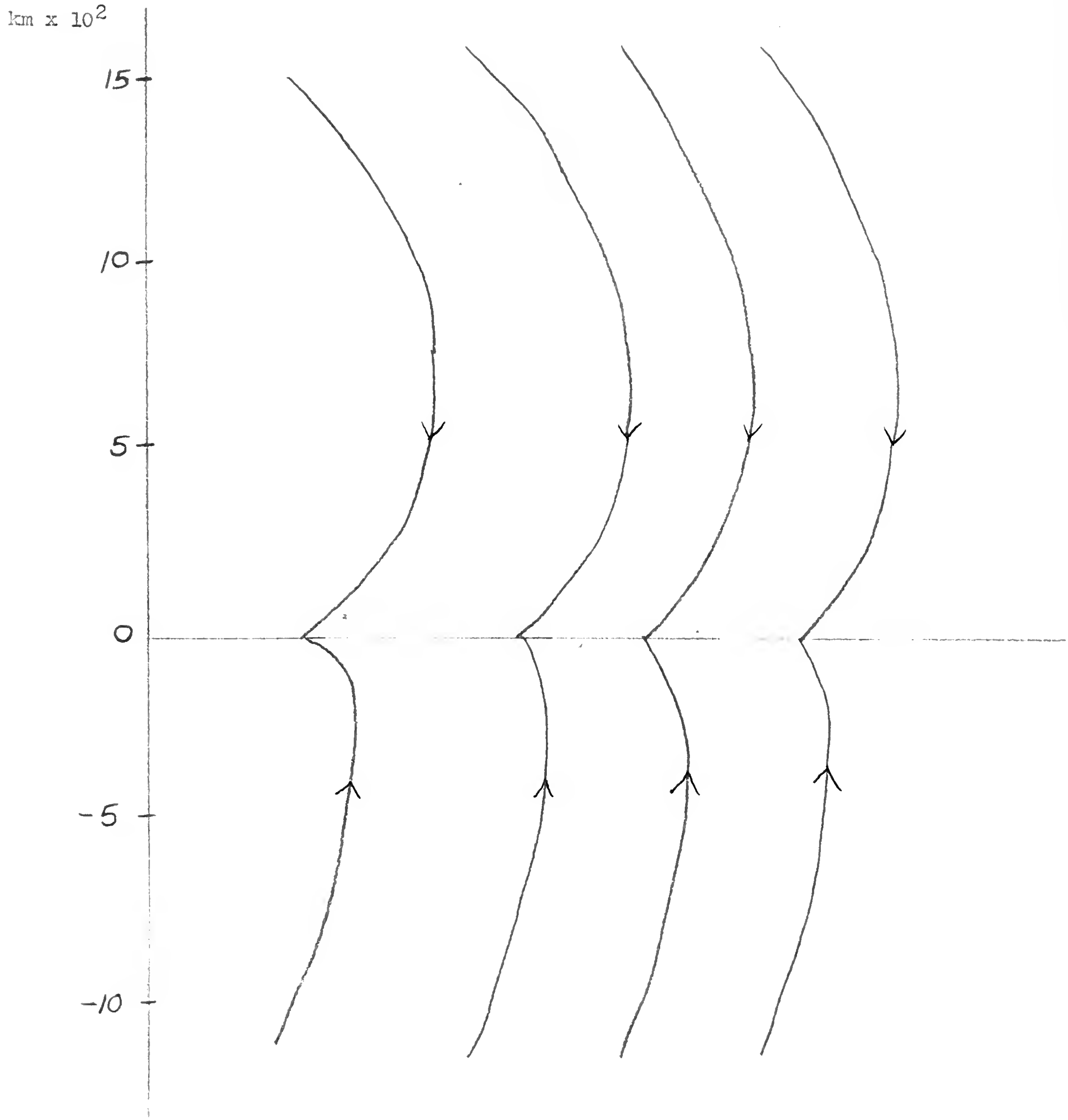
Sea level streamlines for
$u_o = -5$ mps, $v_o = 0$ at $y = -500$ km

Fig. 3



Sea level streamlines for
u_o = -5 mps, v_o = -2 mps at y = 500 km

Fig. 4



Sea level streamlines for
u_o = -5 mps, v_o = -5 mps at y = 0

Fig. 5

III. SPATIALLY INDEPENDENT MOTION WITH VARIABLE CORIOLIS PARAMETER

In this approach Forsdyke's [4] assumptions were used but the Coriolis parameter was varied to secure a more general solution. Motions were considered uniform which is probably valid over areas five degrees square. The pressure distribution $p = p_0 + p_1 x \sin \frac{2\pi t}{T}$ was assumed.

The equations of motion with the above pressure distribution are:

$$\frac{du}{dt} = \lambda v - \frac{1}{\rho}\frac{\partial p}{\partial x} = \lambda v - \frac{p_1}{\rho} \sin \frac{2\pi t}{T} \tag{7}$$

$$\frac{dv}{dt} = -\lambda u \tag{8}$$

Differentiating (8) with respect to t, we obtain

$$\frac{d^2 v}{dt^2} = -\lambda \frac{du}{dt} - u\frac{d\lambda}{dt} \tag{9}$$

moreover $\lambda = 2\omega \sin\phi$; $\frac{d\lambda}{dt} = 2\omega \cos\phi \frac{d\phi}{dt}$; $\phi = y/a$; and

$\frac{d\phi}{dt} = \frac{1}{a}\ dy/dt = v/a$. Since $\cos\phi$ varies only 20% from 0° to 20°, we assume $\cos\phi = 1$ and it follows that,

$$\frac{d\lambda}{dt} = \frac{2\omega v}{a} \tag{10}$$

Substituting the expressions for du/dt, u, and $d\lambda/dt$ from equations (7), (8), and (10) respectively, we get

$$\frac{d^2v}{dt^2} - \frac{2\omega v}{a\lambda}\frac{dv}{dt} + \lambda^2 v = \frac{\lambda P_1}{\rho} \sin\frac{2\pi t}{T} \qquad (11)$$

To solve this differential equation a mean value of $v(\bar{v})$ was assumed giving a second order equation with constant coefficients.

The solution with constants of integration evaluated for the boundary condition, $v = 0$ at $t = 0$ and $C_1 = 1$ is

$$v = e^{\frac{\omega\bar{v} + \sqrt{\omega^2\bar{v}^2 - a^2\lambda^4}}{a\lambda}t} + (-1-B)\, e^{\frac{\omega\bar{v} - \sqrt{\omega^2\bar{v}^2 - a^2\lambda^4}}{a\lambda}t} + A\sin\frac{2\pi t}{T} + B\cos\frac{2\pi t}{T}$$

$$u = -\left(\frac{\omega\bar{v} + \sqrt{\omega^2\bar{v}^2 - a^2\lambda^4}}{a\lambda^2}\right) e^{\frac{\omega\bar{v} - \sqrt{\omega^2\bar{v}^2 - a^2\lambda^4}}{a\lambda}t} + (1+B)\left(\frac{\omega\bar{v} - \sqrt{\omega^2\bar{v}^2 - a^2\lambda^4}}{a\lambda^2}\right) e^{\frac{\omega\bar{v} - \sqrt{\omega^2\bar{v}^2 - a^2\lambda^4}}{a\lambda}t} - \frac{A}{\lambda}\frac{2\pi}{T}\cos\frac{2\pi t}{T} + \frac{B}{\lambda}\frac{2\pi}{T}\sin\frac{2\pi t}{T}$$

where $$A = \frac{\lambda P_1(\lambda^2T^2 - 4\pi^2)}{\rho T^2\left[\left(\frac{\lambda^2T^2 - 4\pi^2}{T^2}\right)^2 + \left(\frac{4\pi\omega\bar{v}}{a\lambda T}\right)^2\right]}$$

and $$B = \frac{P_1 4\pi\omega\bar{v}}{\rho a T\left[\left(\frac{\lambda^2T^2 - 4\pi^2}{T^2}\right)^2 + \left(\frac{4\pi\omega\bar{v}}{a\lambda T}\right)^2\right]}$$

The special case for which $\bar{v} = 0$ gives Forsdyke's solution. Assigning normal numerical values led to imaginary values of u and v and hence the analysis was carried no further. Some numerical constants will lead to real values for u and v; however, the results do not conform to observed patterns.

IV. SPATIALLY INDEPENDENT MOTION FOR A CERTAIN PRESSURE DISTRIBUTION

Forsdyke [4] suggested a pressure distribution of the form $p = p_0 + Ae^{-\epsilon t} x + B(1-e^{-\gamma t})y$ – where A, B, ϵ, γ are constants to be applied to the spatially independent constant Coriolis parameter solution. He states this solution might be applied to curved isobars.

This statement appears to be inconsistent for at any time t, the pressure is a linear function of x and y giving an isobaric pattern consisting of straight parallel lines with their orientation changing with time. At t = 0 the isobars are a function of x and perpendicular to the equator. As t increases and approaches infinity the isobars rotate and become parallel to the equator and are functions of y alone.

The constants A and B are the pressure gradients at times t = 0 and t = ∞ respectively. The constants ϵ and γ determine how rapidly the pressure gradients change in the x and y directions.

In applying this pressure distribution ϵ and γ were made equal for simplification.

The equations of motion with Forsdyke's assumption and the above pressure distribution become

$$\frac{du}{dt} = \lambda v - \frac{A}{\rho} e^{-\gamma t} \tag{12}$$

$$\frac{dv}{dt} = -\lambda u - \frac{B}{\rho}(1-e^{-\gamma t}) \tag{13}$$

The general solution of this set of differential equations consists of the complementary functions

$$u = C_2 \sin\lambda t - C_1 \cos\lambda t$$

$$v = C_1 \sin\lambda t + C_2 \cos\lambda t$$

and the particular integrals

$$u = \frac{B}{\rho\lambda} - \frac{(B\lambda + A\gamma)}{\rho(\gamma^2+\lambda^2)} e^{-\gamma t}$$

$$v = \frac{A\lambda - B\gamma}{\rho(\gamma^2+\lambda^2)} e^{-\gamma t}$$

If C_1 and C_2 are evaluated at the boundary conditions $u = u_o$, $v = v_o$ at $t = 0$ and the values of geostrophic wind

$$u_g = -\frac{1}{\rho\lambda}\frac{\partial p}{\partial y} = -\frac{B}{\rho\lambda}(1 - e^{-\gamma t})$$

$$v_g = \frac{1}{\rho\lambda}\frac{\partial p}{\partial x} = \frac{A}{\rho\lambda} e^{-\gamma t}$$

are inserted we have

$$u = \left[\frac{B\gamma - A\lambda}{\rho(\gamma^2+\lambda^2)} + v_o\right]\sin\lambda t + \left[u_o - \frac{B\lambda}{\rho} + \frac{B\gamma - A\lambda}{\rho(\gamma^2+\lambda^2)}\right]\cos\lambda t \qquad (14)$$

$$+ \frac{B}{\rho\lambda} - \left(\frac{\lambda^2}{\gamma^2+\lambda^2}\right)\left(u_g + \frac{B}{\rho\lambda}\right) - \frac{\gamma\lambda}{\gamma^2+\lambda^2} v_g$$

$$v = \left[\frac{B}{\rho\lambda} - \frac{B\lambda - A\gamma}{\rho(\gamma^2+\lambda^2)} - u_o\right]\sin\lambda t + \left[v_o + \frac{B\gamma - A\lambda}{\rho(\gamma^2+\lambda^2)}\right]\cos\lambda t$$

$$-\left(\frac{\gamma\lambda}{\gamma^2+\lambda^2}\right)\left(u_g + \frac{B}{\rho\lambda}\right) + \frac{\lambda^2}{\gamma^2+\lambda^2} v_g \qquad (15)$$

If A = 0 and $\gamma = \infty$ the pressure distribution becomes $p = p_0 + By$, which is the same as used in Chapter II of this paper and also in one of Forsdyke's [4] solutions for a particular time variation of pressure.

Substituting A = 0 and $\gamma = \infty$ in (14) and (15) we get

$$u = v_0 \sin \lambda t + \left(u_0 - \frac{B\lambda}{\rho}\right) \cos \lambda t \quad \frac{B}{\rho\lambda} \tag{16}$$

$$v = \left(\frac{B}{\rho\lambda} - u_0\right) \sin \lambda t + v_0 \cos \lambda t \tag{17}$$

Equations (16) and (17) are the same as Forsdyke obtained when the substitutions $u^1 = u_0 - u_g$ and $v^1 = v_0 - v_g$ are made since $u_g = \frac{-B\lambda}{\rho}$ and $v_g = 0$ for A = o, $\gamma = \infty$.

Equations (16) and (17) differ from (3) and (4) in spite of the same pressure distribution. This is consistent since different restrictions were placed on the motion; namely, constant Coriolis parameter and independence of space for (16) and (17), while steady state, independence of longitude, and a variable Coriolis parameter in (3) and (4).

Numerical evaluation of equations (16) and (17) could be done by assigning values to the constants A, B, and γ consistent with observed pressure patterns and then computing u and v as a function of latitudes and time. This evaluation was not undertaken here for lack of time.

V. CONCLUSIONS

The equations of motion may be solved for equatorial regions only with rather restrictive assumptions even for relatively simple pressure fields.

Although the solutions in this paper give velocity fields resembling observed mean motions, and one commonly used restriction has been lifted, namely, the requirement non-divergence, they appear to be of little practical value in routine weather analysis. In addition to other limitations the necessary computations are laborious and time consuming.

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